A new class of non-aligned Einstein-Maxwell solutions with a geodesic, shearfree and non-expanding multiple Debever-Penrose vector

Norbert Van den Bergh

Ghent University, Department of Mathematical Analysis FEA16,
Galglaan 2, 9000 Ghent, Belgium

E-mail: `norbert.vandenbergh@ugent.be`

Abstract. In a recent study of algebraically special Einstein-Maxwell fields[1] it was shown that, for non-zero cosmological constant, non-aligned solutions cannot have a geodesic and shearfree multiple Debever-Penrose vector $\boldsymbol{k}$. When $\Lambda = 0$ such solutions do exist and can be classified, after fixing the null-tetrad such that $\Psi_0 = \Psi_1 = \Phi_1 = 0$ and $\Phi_0 = 1$, according to whether the Newman-Penrose coefficient π is 0 or not. The family $\pi = 0$ contains the Griffiths solutions[2], with as sub-families the Cahen-Spelkens, Cahen-Leroy and Szekeres metrics. It was claimed in [2] and repeated in [1] that for $\pi = 0$ both null-rays $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are non-twisting ($\bar{\rho} - \rho = \bar{\mu} - \mu = 0$): while it is certainly true that $\mu(\bar{\rho} - \rho) = 0$, the case $\mu = 0$ appears to have been overlooked. A family of solutions is presented in which $\boldsymbol{k}$ is twisting but non-expanding.

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1. Introduction

In the quest for exact solutions of the Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{1}$$

a large amount of research has been devoted to the study of *aligned* Einstein-Maxwell fields, in which at least one of the principal null directions (PND's) of the electromagnetic field tensor $\boldsymbol{F}$ is parallel to a PND of the Weyl tensor, a so called Debever-Penrose direction, see [6] and citations therein. A systematic attempt at classifying the algebraically special *non-aligned* solutions was initiated in [1]. One of the topics considered, dealing with the reverse of the Goldberg-Sachs theorem, enquired after the existence of algebraically special (non-conformally flat and non-null) Einstein-Maxwell fields with a possible non-zero cosmological constant for which the multiple Weyl-PND $\boldsymbol{k}$ is geodesic and shear-free‡ ($\Psi_0 = \Psi_1 = \kappa = \sigma = 0$) and for which $\boldsymbol{k}$ is *not* parallel to a PND of $\boldsymbol{F}$ ($\Phi_0 \neq 0$). In order to avoid frequent referring to the equations of [1] I present the commutator relations, GHP,

‡ Throughout I use the sign conventions and notations of [6] §7.4, with the tetrad basis vectors taken as $\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \overline{\boldsymbol{m}}$ with $-k^a\ell_a = 1 = m^a\overline{m}_a$. When using the Geroch-Held-Penrose formalism, I will write primed variables, such as κ', σ', ρ' and τ', as their Newman-Penrose equivalents $-\nu, -\lambda, -\mu$ and $-\pi$.

Bianchi and Maxwell equations in the Appendix and I repeat part of the reasoning of [1]: choosing a null-rotation about $\boldsymbol{k}$ such that $\Phi_1 = 0$, it follows that $\Phi_2 \neq 0$§. Using the GHP formalism the Maxwell equations (64,65) and Bianchi equations (66-69) become then

$$ð\Phi_0 = 0, \tag{2}$$
$$ð'\Phi_0 = -\pi\Phi_0, \tag{3}$$
$$Þ\Phi_0 = 0, \tag{4}$$
$$Þ'\Phi_0 = -\mu\Phi_0, \tag{5}$$
$$ð\Phi_2 = -\nu\Phi_0 + \tau\Phi_2, \tag{6}$$
$$Þ\Phi_2 = -\lambda\Phi_0 + \rho\Phi_2, \tag{7}$$
$$ð\Psi_2 = -\pi\Phi_0\overline{\Phi_2} + 3\tau\Psi_2, \tag{8}$$
$$Þ\Psi_2 = \mu|\Phi_0|^2 + 3\rho\Psi_2, \tag{9}$$

after which the commutators $[ð', ð], [ð', Þ], [ð, Þ']$ and $[Þ', Þ]$ applied to Φ_0 give

$$ð\pi = (3\rho - \overline{\rho})\mu - 2\Psi_2 + \frac{R}{12}, \tag{10}$$
$$Þ\pi = 3\rho\pi, \tag{11}$$
$$ð\mu = \overline{\lambda}\pi + 3\mu\tau, \tag{12}$$
$$Þ\mu = \pi(\overline{\pi} + 3\tau) + 2\Psi_2 - \frac{R}{12}. \tag{13}$$

Herewith GHP equation (63') becomes a simple algebraic equation for Ψ_2,

$$\Psi_2 = \rho\mu - \tau\pi + \frac{R}{12}, \tag{14}$$

the Þ derivative of which, using (11,13,57,59), results in $\rho R = 0$.

As $\rho = 0$ would imply $\Phi_0 = 0$, it follows that an algebraically special Einstein-Maxwell solution possessing a shear-free and geodesic multiple Debever-Penrose vector, which is not a PND of $\boldsymbol{F}$, necessarily has a vanishing cosmological constant. The corresponding class of solutions is non-empty, as it contains the Griffiths metrics[2], encompassing as special cases the metrics of [7, 8, 9, 10].

In [2] Griffiths claimed that for $\pi = 0$ both null-rays $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are necessarily non-twisting ($\bar{\rho} - \rho = \bar{\mu} - \mu = 0$). As a consequence it was also claimed in [1] that the Griffiths metrics are uniquely characterised by the condition $\pi = 0$. However, when $\pi = 0$ the only conclusion to be drawn from (10, 14) is that $\mu(\bar{\rho} - \rho) = 0$. When ρ is real this indeed leads to the metrics of [2], but the case $\mu = 0$ appears to have been overlooked and leads, as shown in the section below, to new classes of solutions‖.

§ with $\Phi_2 = 0$ ℓ becomes geodesic and shear-free and the Goldberg-Sachs theorem implies $\Psi_3 = \Psi_4 = 0$. The Petrov type would then be D, in which case [4, 5] the only null Einstein-Maxwell solutions are given by the (doubly aligned) Robinson-Trautman metrics.

‖ The case $\mu = 0$ is not to be regarded as a Kundt family, as the null ray generated by ℓ is neither geodesic nor shear-free.

2. The twisting and non-expanding family

When $\pi=0$ and $\mu=0$ the equations of the previous paragraph immediately imply $\Psi_0=\Psi_1=\Psi_2=0$ and $\Psi_3=\rho\nu-\lambda\tau$. As little progress appears to be possible in the general case, I restrict to solutions for which $\boldsymbol{k}$ is non-expanding ($\rho+\overline{\rho}=0$). Acting on this condition with the ð and Þ operators, the GHP equations yield $\tau=0$ and

$$\rho^2+|\Phi_0|^2=0, \tag{15}$$

the ð derivative of which implies $\lambda=\Phi_2\overline{\Phi_0}\rho^{-1}$. Translating these results into Newman-Penrose language and fixing a boost and spatial rotation in the $\boldsymbol{k},\boldsymbol{\ell}$ and $\boldsymbol{m},\overline{\boldsymbol{m}}$ planes such that $\Phi_0=1$ and $\rho=i$, it follows that the only non-0 spin coefficients are ρ, ν and $\lambda=-i\Phi_2$, with the only non-vanishing components of the Weyl spinor being $\Psi_3=i\nu$ and Ψ_4. As $[D,\Delta]=0$ coordinates u,v and $\zeta,\overline{\zeta}$ exist such that $D=\partial_u$, $\Delta=\partial_v$ and

$$\delta=e^{-iu}(\xi\partial_\zeta+\eta\partial_{\overline{\zeta}}+P\partial_u+Q\partial_v), \tag{16}$$

ξ,η,P,Q being arbitrary functions. The e^{-iu} factor is included for convenience: applying the $[\delta,D]$ commutator to u,v and ζ shows that ξ,η,P,Q are functions of $v,\zeta,\overline{\zeta}$ only. Introducing new variables $n=e^{-iu}\nu$ and $\phi=e^{-2iu}\Phi_2$ it follows that also n and ϕ depend on $v,\zeta,\overline{\zeta}$ only, with the full set of Jacobi and field equations reducing to the following system of pde's:

$$P_v+i\overline{P}\,\overline{\phi}-\overline{n}=0, \tag{17}$$
$$Q_v+i\overline{Q}\,\overline{\phi}=0, \tag{18}$$
$$\xi_v+i\overline{\eta}\,\overline{\phi}=0, \tag{19}$$
$$\eta_v+i\overline{\xi}\,\overline{\phi}=0, \tag{20}$$

$$e^{-iu}\overline{\delta}P-e^{iu}\delta\overline{P}-2i|P|^2=0, \tag{21}$$
$$e^{-iu}\overline{\delta}Q-e^{iu}\delta\overline{Q}-2i\Re(Q\overline{P}-1)=0, \tag{22}$$
$$e^{-iu}\overline{\delta}\xi-e^{iu}\delta\overline{\eta}-i(\xi\overline{P}+\overline{\eta}P)=0, \tag{23}$$

$$e^{iu}\delta n=-iPn+2|\phi|^2, \tag{24}$$
$$e^{iu}\delta\phi=-2iP\phi-n, \tag{25}$$

with the Ψ_4 component of the Weyl spinor given by $\Psi_4=ie^{2iu}(\overline{n}\overline{P}+\Delta\phi)+e^{iu}\overline{\delta}n$.

3. The case $\phi=\phi(\zeta,\overline{\zeta})$

Assuming $\phi=\phi(\zeta,\overline{\zeta})$, writing $\phi=H^2h^2$ with $H>0$ and $|h|=1$, (18,19,20) integrate to $Q=q_1e^{H^2v}+q_2e^{-H^2v}$, $\xi=\overline{c_1}e^{H^2v}+\overline{c_2}e^{-H^2v}$, $\eta=ih^{-2}(-c_1e^{H^2v}+c_2e^{-H^2v})$ with q_A,c_A depending on $\zeta,\overline{\zeta}$ only and $q_J+ih^{-2}\overline{q_J}=0$ ($J=1,2$). A coordinate transformation $\zeta\to\tilde{\zeta}(\zeta,\overline{\zeta})$ allows one then to put (writing $\tilde{\zeta}=x+iy$ and re-defining q_J),

$$\delta=e^{-iu}[P\partial_u+e^{H^2v-C_1-i\frac{\pi}{4}}h^{-1}(\partial_x+q_1\partial_v)+e^{-H^2v-C_2+i\frac{\pi}{4}}h^{-1}(\partial_y+q_2\partial_v)], \tag{26}$$

with C_J and q_J real functions of x and y.

An expression for P is obtained from (23),

$$P = e^{i\frac{\pi}{4}} h^{-1}[e^{H^2 v - C_1}(\frac{h_{,x}}{h} - C_{2,x} - v(H^2)_{,x} - q_1 H^2) \\ + i e^{-H^2 v - C_2}(\frac{h_{,y}}{h} - C_{1,y} + v(H^2)_{,y} + q_2 H^2)], \tag{27}$$

after which n follows from (17),

$$n = -2hH^2\{e^{H^2 v - C_1 - i\frac{\pi}{4}}[(1+2vH^2)\frac{H_{,x}}{H} + C_{2,x} + q_1 H^2] \\ + e^{-H^2 v - C_2 + i\frac{\pi}{4}}[(1-2vH^2)\frac{H_{,y}}{H} + C_{1,y} - q_2 H^2]\}. \tag{28}$$

Herewith (21) becomes a polynomial identity in powers of v and $e^{H^2 v}$,

$$v^2 e^{2H^2 v} H_{,x}^2 - v^2 e^{-2H^2 v} H_{,y}^2 + \ldots = 0, \tag{29}$$

showing that H is necessarily constant.

Introducing new variables $r_1 = -H^2 q_1 - C_{2,x}$, $r_2 = H^2 q_2 - C_{1,y}$ the remaining coefficients of (29) lead to the equations,

$$r_{1,x} - 2r_1^2 - r_1(C_1 + C_2)_{,x} = 0, \tag{30}$$

$$r_{2,y} - 2r_2^2 - r_2(C_1 + C_2)_{,y} = 0. \tag{31}$$

Substituting this in (28), equation (24) becomes a Liouville equation determining $C_1 + C_2$,

$$(C_1 + C_2)_{,xy} + 2H^2 e^{C_1 + C_2} = 0, \tag{32}$$

while (25) reduces to an identity. A final equation is (22), which now becomes

$$r_{2,x} + r_{1,y} - H^2 e^{C_1 + C_2} = 0. \tag{33}$$

The general solution of the Liouville equation being given by

$$e^{C_1 + C_2} = -\frac{a_{,x} b_{,y}}{H^2 (a+b)^2}, \tag{34}$$

($a = a(x)$ and $b = b(y)$ arbitrary functions), r_1 and r_2 are given by (30,31) as,

$$r_1 = \frac{b_{,y}}{2(a+b)(1 + A(a+b))},\ r_2 = \frac{a_{,x}}{2(a+b)(1 + B(a+b))}, \tag{35}$$

with arbitrary functions $A = A(x), B = B(y)$. Herewith (33) reduces to the condition

$$(\frac{\mathrm{d}A}{\mathrm{d}a} - A^2)(1 + B(a+b))^2 + (\frac{\mathrm{d}B}{\mathrm{d}b} - B^2)(1 + A(a+b))^2 = 0, \tag{36}$$

implying either $A_{,a} - A^2 = B_{,b} - B^2 = 0$, or $\log\frac{1+B(a+b)}{1+A(a+b)}$ being separable in x and y. As the latter condition again can be shown to imply $A_{,a} - A^2 = B_{,b} - B^2 = 0$, we conclude that the general solution is given by

$$r_1 = \frac{a_{,x}}{2(a+b)} \frac{k - k_0 b}{k + k_0 a},\ r_2 = \frac{b_{,y}}{2(a+b)} \frac{k - k_0 a}{k + k_0 b}, \tag{37}$$

with either $k_0 = 1$ and k an arbitrary (real) constant¶, or $k_0 = 0, k = 1$ (corresponding to the special case $A = B = 0$).

¶ which can be taken to be 0 or 1

The resulting metric appears to contain two arbitrary functions, being the phase factor $h(x,y)$ of ϕ and the function $F(x,y)$ defined by $e^{C_1-C_2}=-\frac{a_{,x}}{b_{,y}}e^{2F}$. These however can be eliminated by the coordinate transformations $u\to u-i\log h$ and $v\to H^2v-F$, after which the dual basis takes the form,

$$\boldsymbol{\omega}^1=\frac{e^{iu}}{2H(a+b)}(e^{i\frac{\pi}{4}-v}\mathrm{d}a-e^{-i\frac{\pi}{4}+v}\mathrm{d}b), \tag{38}$$

$$\boldsymbol{\omega}^3=\frac{1}{H^2}\mathrm{d}v-\frac{1}{H^2(a+b)}[\frac{(2a+b)k_0+k}{2(k_0a+k)}\mathrm{d}a-\frac{(a+2b)k_0+k}{2(k_0b+k)}\mathrm{d}b], \tag{39}$$

$$\boldsymbol{\omega}^4=\mathrm{d}u+\frac{1}{2(a+b)}[\frac{k_0a-k}{k_0b+k}e^{-2v}\mathrm{d}a-\frac{k_0b-k}{k_0a+k}e^{2v}\mathrm{d}b], \tag{40}$$

$(k_0,k)=(1,0)$, $(1,1)$ or $(0,1)$.

4. Discussion

The null tetrad (38-40) determines a (presumably) new family of Einstein-Maxwell solutions with zero cosmological constant and with Maxwell field and energy-momentum tensor given by

$$\boldsymbol{F}=iH^2(\boldsymbol{\omega}^1-\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^3+i(e^{-2iu}\boldsymbol{\omega}^1-e^{2iu}\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^4, \tag{41}$$

$$\boldsymbol{T}=2H^2(e^{-2iu}\boldsymbol{\omega}^1\otimes\boldsymbol{\omega}^1+e^{2iu}\boldsymbol{\omega}^2\otimes\boldsymbol{\omega}^2+H^2\boldsymbol{\omega}^3\otimes\boldsymbol{\omega}^3)+2\boldsymbol{\omega}^4\otimes\boldsymbol{\omega}^4. \tag{42}$$

The Petrov type is III, with the multiple Debever-Penrose vector $\boldsymbol{k}=\partial_u$ being geodesic, shear-free and twisting but non-expanding. The real null vector $\boldsymbol{\ell}$, fixed by a null-rotation such that $\Phi_1=0$, is non-diverging, but is non-geodesic and has non-vanishing shear. It follows that figure (2) in [1] has to be amended as in Fig. 1 below.

For all solutions ∂_u is clearly a null Killing vector. While in general (i.e. with k and $k_0\neq 0$) the isometry group is 2-dimensional, with the second Killing vector given by

$$\boldsymbol{K}_2=k_0^2(a+b)\partial_v+(a^2k_0^2-k^2)\partial_a-(b^2k_0^2-k^2)\partial_b, \tag{43}$$

the special cases $k_0=0,k=1$ and $k_0=1,k=0$ admit a 3-dimensional group of isometries, with third Killing vector

$$\boldsymbol{K}_3=a\partial_a+b\partial_b. \tag{44}$$

In the latter cases the isometry group has Bianchi type III, with the orbits being time-like hypersurfaces parametrized by the null coordinate v. For the case $k_0=0,k=1$ the tetrad simplifies to

$$\boldsymbol{\omega}^1=\frac{e^{iu}}{2H(a+b)}[e^{i\frac{\pi}{4}-v}\mathrm{d}a-e^{-i\frac{\pi}{4}+v}\mathrm{d}b], \tag{45}$$

$$\boldsymbol{\omega}^3=\frac{1}{H^2}[\mathrm{d}v-\frac{1}{2(a+b)}\mathrm{d}(a-b)], \tag{46}$$

$$\boldsymbol{\omega}^4=\mathrm{d}u-\frac{1}{2(a+b)}[e^{-2v}\mathrm{d}a-e^{2v}\mathrm{d}b] \tag{47}$$

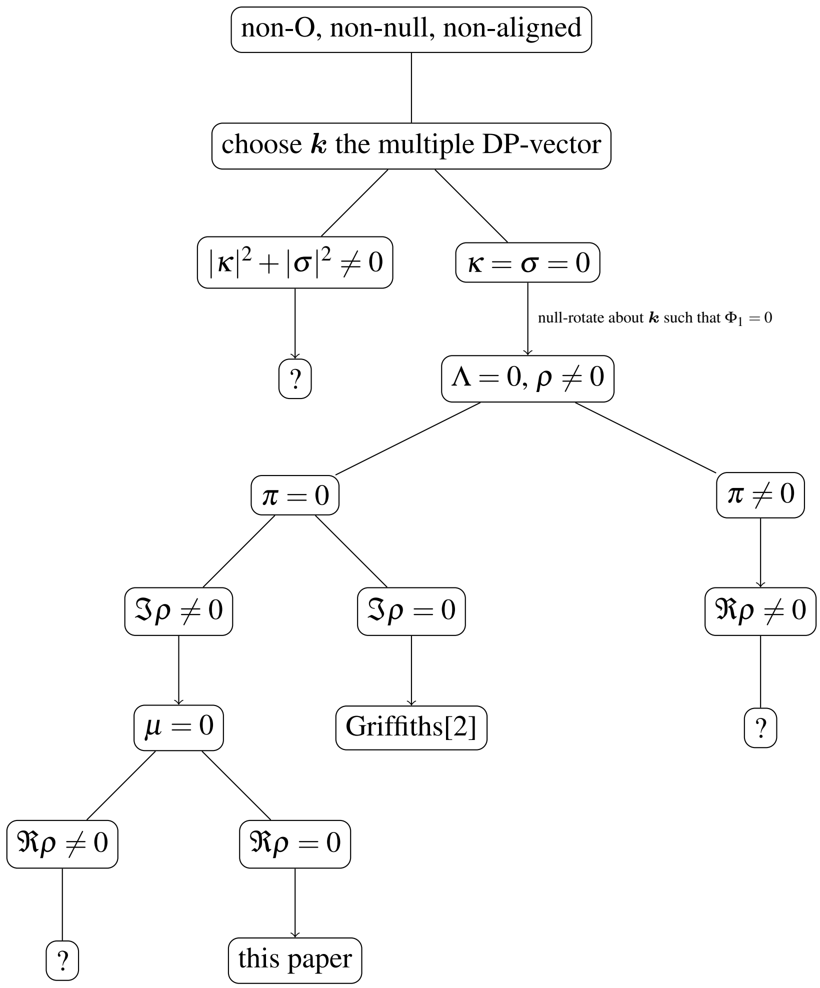

Figure 1. Algebraically special non-nul Einstein-Maxwell solutions for which the multiple Weyl-PND $\boldsymbol{k}$ is not a PND of $\boldsymbol{F}$.

and the line-element becomes

$$\mathrm{d}s^2=\frac{1}{H^2}[-2\mathrm{d}v+\frac{1}{a+b}\mathrm{d}(a-b)]\mathrm{d}u+\frac{1}{H^2(a+b)}(e^{-2v}\mathrm{d}a-e^{2v}\mathrm{d}b)\mathrm{d}v+\frac{\cosh 2v}{H^2(a+b)^2}\mathrm{d}a\mathrm{d}b. \quad (48)$$

The non-0 components of the Weyl-spinor are then given by

$$\Psi_3=-H^3e^{i(u-\frac{\pi}{4})}(e^v+ie^{-v}) \quad (49)$$

$$\Psi_4=2H^4e^{2iu}\cosh 2v. \quad (50)$$

All the Carminati-McLenaghan invariants are regular functions of the essential coordinate v over the interval $]-\infty,+\infty[$. The same holds for the case $k_0=1, k=0$ (in which the essential coordinate is $v+\log(b/a)$), but although all the (CM-) invariants are transformed into each other under the coordinate transformation $v\to v+\log(b/a)$ the two special cases are inequivalent.

∂_u being Killing vector, it might look peculiar that the Weyl spinor components Ψ_3 and Ψ_4 still depend on u, even though the frame was "invariantly" fixed. This is due to the fixation having been done by means of a null rotation putting $\Phi_0=1$: the resulting frame scalars are then not genuine Cartan invariants and, as the Maxwell field itself does not inherit the space-time symmetries ($\boldsymbol{F}$ is not Lie-propagated along the integral curves of the null Killing vector ∂_u), the frame scalars depend on u as well. This remark also shows that the present solutions are distinct from the Lucács et al. solutions admitting null-Killing vectors [11], as there the Maxwell field does inherit all the space-time symmetries.

Appendix A

Weights[+] of the spin-coefficients, the Maxwell and Weyl spinor components and the GHP operators:

$$\begin{aligned}
&\kappa:[3,1],\nu:[-3,-1],\sigma:[3,-1],\lambda:[-3,1],\\
&\rho:[1,1],\mu:[-1,-1],\tau:[1,-1],\pi:[-1,1],\\
&\Phi_0:[2,0],\Phi_1:[0,0],\Phi_2:[-2,0],\\
&\Psi_0:[4,0],\Psi_1:[2,0],\Psi_2:[0,0],\Psi_3:[-2,0],\Psi_4:[-4,0],\\
&\eth:[1,-1],\eth':[-1,1],\text{Þ}':[-1,-1],\text{Þ}:[1,1].
\end{aligned}$$

The prime operation is an involution with

$$\kappa'=-\nu,\sigma'=-\lambda,\rho'=-\mu,\tau'=-\pi, \tag{51}$$

$$\Psi_0{}'=\Psi_4,\Psi_1{}'=\Psi_3,\Psi_2{}'=\Psi_2, \tag{52}$$

$$\Phi_0'=-\Phi_2,\Phi_1'=-\Phi_1. \tag{53}$$

The GHP commutators acting on (p,q) weighted quantities are given by:

$$\begin{aligned}
[\text{Þ},\text{Þ}'] &= (\pi+\overline{\tau})\eth+(\overline{\pi}+\tau)\eth'+(\kappa\nu-\pi\tau+\frac{R}{24}-\Phi_{11}-\Psi_2)p\\
&\quad+(\overline{\kappa\nu}-\overline{\pi\tau}+\frac{R}{24}-\Phi_{11}-\overline{P}_2)q,
\end{aligned}\tag{54}$$

$$\begin{aligned}
[\eth,\eth'] &= (\mu-\overline{\mu})\text{Þ}+(\rho-\overline{\rho})\text{Þ}'+(\lambda\sigma-\mu\rho-\frac{R}{24}-\Phi_{11}+\Psi_2)p\\
&\quad-(\overline{\lambda\sigma}-\overline{\mu\rho}-\frac{R}{24}-\Phi_{11}+\overline{P}_2)q,
\end{aligned}\tag{55}$$

$$\begin{aligned}
[\text{Þ},\eth] &= \overline{\pi}\,\text{Þ}-\kappa\text{Þ}'+\overline{\rho}\,\eth+\sigma\eth'+(\kappa\mu-\sigma\pi-\Psi_1)p\\
&\quad+(\overline{\kappa\lambda}-\overline{\pi\rho}-\Phi_{01})q.
\end{aligned}\tag{56}$$

GHP equations:

$$\text{Þ}\rho-\eth'\kappa=\rho^2+\sigma\overline{\sigma}-\overline{\kappa}\tau+\kappa\pi+\Phi_{00}, \tag{57}$$

$$\text{Þ}\sigma-\eth\kappa=(\rho+\overline{\rho})\sigma+(\overline{\pi}-\tau)\kappa+\Psi_0, \tag{58}$$

$$\text{Þ}\tau-\text{Þ}'\kappa=(\tau+\overline{\pi})\rho+(\overline{\tau}+\pi)\sigma+\Phi_{01}+\Psi_1, \tag{59}$$

$$\text{Þ}\nu-\text{Þ}'\pi=(\pi+\overline{\tau})\mu+(\overline{\pi}+\tau)\lambda+\Psi_3+\overline{\Phi_1}\Phi_2, \tag{60}$$

$$\eth\rho-\eth'\sigma=(\rho-\overline{\rho})\tau+(\mu-\overline{\mu})\kappa+\Phi_{01}-\Psi_1, \tag{61}$$

[+] Objects x transforming under boosts and rotations as $x\rightarrow A^{\frac{p+q}{2}}e^{i\frac{p-q}{2}\theta}x$ are called *well-weighted of type* (p,q).

$$Þ'\sigma - ð\tau = -\sigma\mu - \overline{\lambda}\rho - \tau^2 + \kappa\overline{\nu} - \Phi_{02}, \tag{62}$$
$$Þ'\rho - ð'\tau = -\overline{\mu}\rho - \lambda\sigma - \tau\overline{\tau} + \kappa\nu - \frac{R}{12} - \Psi_2. \tag{63}$$

Maxwell equations:

$$Þ\Phi_1 - ð'\Phi_0 = \pi\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2, \tag{64}$$
$$Þ\Phi_2 - ð'\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + \rho\Phi_2. \tag{65}$$

Bianchi equations (with $\Phi_{IJ} = \Phi_I\overline{\Phi_J}$ and $\Lambda = R/4 = constant$):

$$\begin{aligned} ð'\Psi_0 - Þ\Psi_1 + Þ\Phi_{01} - ð\Phi_{00} &= -\pi\Psi_0 - 4\rho\Psi_1 + 3\kappa\Psi_2 + \overline{\pi}\Phi_{00} + 2\overline{\rho}\Phi_{01} + 2\sigma\Phi_{10} \\ &\quad -2\kappa\Phi_{11} - \overline{\kappa}\Phi_{02}, \end{aligned} \tag{66}$$
$$\begin{aligned} Þ'\Psi_0 - ð\Psi_1 + Þ\Phi_{02} - ð\Phi_{01} &= -\mu\Psi_0 - 4\tau\Psi_1 + 3\sigma\Psi_2 - \overline{\lambda}\Phi_{00} + 2\overline{\pi}\Phi_{01} + 2\sigma\Phi_{11} \\ &\quad +\overline{\rho}\Phi_{02} - 2\kappa\Phi_{12}, \end{aligned} \tag{67}$$
$$\begin{aligned} 3ð'\Psi_1 - 3Þ\Psi_2 + 2Þ\Phi_{11} - 2ð\Phi_{10} + ð'\Phi_{01} - Þ'\Phi_{00} &= 3\lambda\Psi_0 - 9\rho\Psi_2 - 6\pi\Psi_1 + 6\kappa\Psi_3 \\ &\quad +(\overline{\mu} - 2\mu)\Phi_{00} + 2(\pi + \overline{\tau})\Phi_{01} + 2(\tau + \overline{\pi})\Phi_{10} + 2(2\overline{\rho} - \rho)\Phi_{11} \\ &\quad +2\sigma\Phi_{20} - \overline{\sigma}\Phi_{02} - 2\overline{\kappa}\Phi_{12} - 2\kappa\Phi_{21}, \end{aligned} \tag{68}$$
$$\begin{aligned} 3Þ'\Psi_1 - 3ð\Psi_2 + 2Þ\Phi_{12} - 2ð\Phi_{11} + ð'\Phi_{02} - Þ'\Phi_{01} &= 3\nu\Psi_0 - 6\mu\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 \\ &\quad -\overline{\nu}\Phi_{00} + 2(\overline{\mu} - \mu)\Phi_{01} - 2\overline{\lambda}\Phi_{10} + 2(\tau + 2\overline{\pi})\Phi_{11} + (2\pi + \overline{\tau})\Phi_{02} \\ &\quad +2(\overline{\rho} - \rho)\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22}. \end{aligned} \tag{69}$$

Acknowledgment

All calculations were done using the Maple symbolic algebra system, while the properties of the metric (48) were checked with the aid of Maple's DifferentialGeometry package[12].

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